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THE NON-LINEAR HEREDITARY-TYPE STRESS–STRAIN RELATION FOR METALS

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Abstract—The stress–strain-time dependance for some metals and alloys based on the idea of non-linear heredity is proposed. In particular the theory can be applied to the metals with delayed yielding (low carbon mild steels). It is shown that the phenomenon of yield delay affects the slope of stress–strain curve in any prescribed loading program. The proposed theory has been applied to the description and analysis of the results of tests of some metals under various loading and unloading rates, and also to the creep and relaxation experiments. Using some additional assumption the basic equation can be applied to the description of material behaviour under repeated loading and to incremental elastic waves propagation.

The problem of the longitudinal waves propagation in non-linear hereditary type medium has been solved using the Laplace transform technique. An asymptotic expantion of the solution in the vicinity of the wave front has been obtained.

. INTRODUCTION

The non-linear-hereditary theory has found wide application for description of time effects in polymers, including reinforced polymers. It turned out that such processes as creep, relaxation, creep recovery may be rather accurately described within a wide range of variation of stress, time and temperature [1–3].

This work is intended to show that the nonlinear-hereditary theory of viscoelasticity may be, with some modifications, applied to the description of the behaviour of metals. It is possible to describe some experimental phenomena which used to be interpreted in different ways and have controversial explanations.

There are known two basic models of metal behaviour which take into account the rate of loading effect. The simplest model according to Karman–Taylor–Rakhmatulin is based on two stress–strain curves, namely static and dynamic. The dynamic diagram is assumed to be independent of the rate of loading. As it turned out, this scheme yields satisfactory results in description of wave propagation in bars. However the principle of differentiation between the "static" and "dynamic" diagrams is not available; moreover, this scheme is unable to describe the observed rate variation of the stress–strain curves.

Another model according to Malvern–Sokolovsky [6, 7], is based on the assumption that there exist a lower static curve which corresponds to loading at an infinitely low rate. The constitutive equation is written as follows:

$$\dot{p} = F[\sigma - \varphi(\varepsilon)], \qquad F(0) = 0. \qquad p = \varepsilon - \sigma/E$$

In this case $\delta = \varphi(\varepsilon)$ is an equation of the stress–strain static curve. Various generalisations and refinements were offered to develop this approach. They bring to good results at the appropriate selection of parameters in any specific case, but, as a rule, turn out to be impractical at other loading processes. Moreover, the conception of static yield stress is rather vague. Usually that is understood as the stress which is measured during ordinary tests on a standard tensile testing machine. However, should the loading rate be some more decreased, the yield stress also decreases [8,9]. Thus, the position of the static curve which should be constructed for loading at infinitely low rate remains to be vague.

There exists one more fact which favoured to attraction of the hereditary theory. This is relationship between yield stress of materials and the history of the loading in the elastic range. Firstly it was observed in experiments with low-carbon steels possessing the effect of the yield delay. The effect of delay at present is studied rather completely (review of works in [10]). Work [11] offers a model of the elastic-plastic medium with delayed yielding; the model is, principally, a generalisation of Taylor–Karman–Rakhmatulin model. The processes of wave propagation in bars which were investigated on the basis of this model, are described in detail in [12]. When constant velocity V_0 is acting at the end of semi-infinite bar, at first an elastic overstress wave spreads through the bar; the rear front of the wave is determined by condition of exhaustion of the yield delay capacity and moves at a velocity of an elastic wave. Since the delay time is the same for all sections, the elastic relief takes place at this front down to stress σ_s, which is conditionally termed the lower yield stress. Strain ε_s corresponding to stress σ_s remains constant during some time, then plastic waves come. This picture was also confirmed in experiments[13]. However the value of ε_s was found to be higher than the strain of the static yield stress. This is not due to the rate effect, because the strain rate in constant value region is equal to zero. But depending on the loading rate, i.e. on the value of the overstress, σ_s (and ε_s) were found to be different. Thus, we have to admit that the history of loading in the elastic range until exhaustion of the yield delay capacity affects the shape of the diagram of the subsequent plastic strain and the value of the yield stress.

A similar effect was discovered in high strain rate testing of a titanium alloy which does not possess the yield delay capacity; it was found that the yield stress depends on tension rate in the elastic range[14].

2. MODEL OF ELASTIC-PLASTIC BODY

As it is known, the hereditary theory of viscoelasticity is based on the hypothesis of existence of the upper instantaneous straining diagram from which "slipping down" takes place with time. This circumstance which allow to eliminate introduction of the static diagram from the constitutive equation and possibility to take into account the elastic strain history gave reason to apply the hereditary theory to behavior of metals[15, 16].

The pecularity of application of the theory to metals consists in replacement the values of total strain in the constitutive equation as suggested in [1, 17] by plastic components

$$\varphi(p) = (1 + K^*)\sigma$$
$$K^*\sigma = \int_0^t K(t-s)\sigma(s)\,ds. \tag{1}$$

Equation $\sigma = \varphi(p)$ determines the instantaneous straining diagram, $p = \varepsilon - \sigma/E$, ε is total strain. The function of $\varphi(p)$ is determined only for the positive values of the argument, $\varphi(0)$ is the dynamic yield stress at an infinitely high strain rate.

Thus, in contrast with scheme of Malvern–Sokolovsky, in a real dynamic process, the time slipping down from the ideal dynamic diagram rather than the rate excess of the static diagram takes place.

The moment of loading beginning in (1) is moment $t = 0$; consequently, the integral term takes into account the entire history of loading in the elastic range. Apparently some internal processes in elastic materials affect the progress of the subsequent plastic strain. At small t the integral in the right-hand part (1) becomes extremely small and transition to the dynamic curve $\sigma = \varphi(\varepsilon - \sigma/E)$ takes place in accordance with the scheme of Taylor–Karman–Rakhmatulin. The eqn (1) is simplest nonlinear hereditary type relationship between values p and σ.

3. ACTIVE LOADING PROCESSES

At first we shall discuss problems, connected with the process of transition from elastic to plastic state. Here, materials possessing the delayed yielding and without it should be discussed separately.

3.1 *Materials without delayed yielding*

If the material does not posses the yield delay capacity, time t_0 of transition to the plastic state is to be found directly from eqn (1); the material remains elastic until

$$\sigma + \int_0^t K(t-s)\sigma(s)\,ds \leq \varphi(0). \tag{2}$$

At the instant t, when the equality becomes valid in (2), the material enters the plastic stage.

The instant and the respective stress are determined as follows. If the machine grips move at a constant velocity, then

$$\sigma/E + \lambda p = Vt, \qquad V = \text{const.} \tag{3}$$

Value λ characterizes the loading system rigidity; at $\lambda = 0$, V is the loading rate, at $\lambda = 1$, V is the total strain rate. In general case $0 < \lambda < 1$. It follows from eqn (1)

$$\varphi(p) = \sigma + \int_0^t K(t-s)(Vs - \lambda p)\,\mathrm{d}s.$$

Let us assume

$$\int_0^t K(t-s)s\,\mathrm{d}s = G_1(t)$$

Then

$$\varphi(p) = \sigma - \lambda \int_0^t K(t-s)p(s)\,\mathrm{d}s + VG_1(t). \tag{4}$$

For function $p(t)$ we obtained a nonlinear integral equation.

The eqn (1) becomes valid beginning from the moment t_0. Therefore, if $t > t_o$, $\varphi(p) > \varphi(0)$; if $t < t_0$, $p \equiv 0$. At $t = t_0$, $p = 0$, $\sigma = \sigma_0 = VEt_0$, eqn (4) also gives the following result

$$\varphi(0) = \sigma_0 + VG_1(\sigma_0/VE). \tag{5}$$

The relation (5) determines yield stress σ_0, i.e. the initial point of the plastic strain curve, corresponding to tension rate V.

As an example Table 1 gives relationship between the yield stress and loading rate for titanium alloy[14] obtained experimentally and calculated from (5); it was assumed that $K(t-s) = k/(t-s)^\alpha$ the material parameters were $\alpha = 0.9$, $k = 0.023$.

To construct the stress-strain curve in plastic range, it is necessary to solve the integral eqn (4). The case when $\lambda = 0$ (constant loading rate) is an exception, then

$$\varphi(p) = \sigma + VG_1(\sigma/VE). \tag{6}$$

The instantaneous curve calculated from the straining diagrams[14] and diagrams of tension at various rates $\dot{\varepsilon}$, are shown in Fig. 1 In the analysis of experimental data with the aid of eqn (1) the stress variation in time was approximated by the piece-wise linear function.

3.2 *Materials with delayed yielding*

Now we discuss the process of transition from elastic to plastic state for materials with yield delay capacity. At present the phenomenon of the yield delay is studied rather completely, despite of that there still exists some misunderstanding in the interpretation of the obtained stress–strain diagrams. As it is known, the yield delay capacity is associated with the extension of the elastic part of the diagram and involves existance of two branches in some region. The value of the upper yield stress is determined by the selected criterion of delay. Transition from

Table 1.

$\dot{\varepsilon}$ 1/s	1.5 10^{-5}	1.8 10^{-3}	5.10^{-1}	5.0
$\sigma_{1\,exp}$, kg/mm^2	31.0	34.2	38.5	40.0
$\sigma_{1\,cal}$, kg/mm^2	30.3	34.8	39.1	40.5

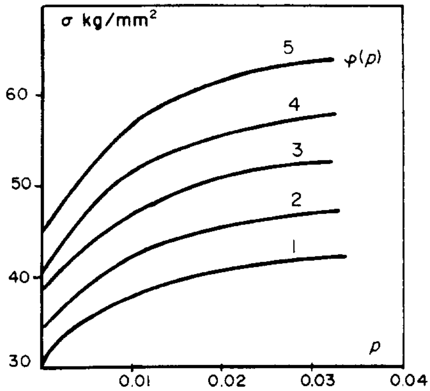

Fig. 1. (1) $\dot{p} = 1.5\ 10^{-5}\ \text{sec}^{-1}$; (2) $\dot{p} = 1.8\ 10^{-3}\ \text{sec}^{-1}$; (3) $\dot{p} = 5.0\ 10^{-1}\ \text{sec}^{-1}$; (4) $\dot{p} = 5.0\ 10^{0}\ \text{sec}^{-1}$.

the elastic to plastic branch may be carried out in different ways and is determined by conditions of the loading process. We shall prove later that the plateau on the material stress–strain diagram appears only in tensioning specimens with a long working part and its length is naturally associated with the specimen length. Similary, under some definite conditions of loading (the stress grows up all the time, for example, according to law $\sigma = \dot{\sigma}t$) the peak stress, i.e. upper yield stress, is not observed. Thus, the obtained stress–strain diagrams should be interpreted with extreme care, paying attention to both the shape of the specimen used and conditions of its loading.

The suggested model just as in [11] permits possibility of the instantaneous variation of the specimen state—appearance of instantaneous plastic strain p and stress drop. However the stress drops not to the static curve as in [11], but to the curve determined by the history of the elastic loading. At the instant t_0 yield delay capacity becomes exhausted

$$\sigma/E + \lambda p = Vt_0 = \sigma_d/E. \tag{7}$$

Since at $t < t_0$, $p = 0$, the eqn (4) has the following form

$$\varphi(p) = \sigma + VG_1(t) - \lambda \int_0^t K(t-s)p(s)\,\mathrm{d}s.$$

At the instant $t = t_0$

$$\varphi(p) = \sigma + VG_1(\sigma_d/VE). \tag{8}$$

Solving (7) and (8) together, we find the values of σ and p at the moment $t = t_0$ after the condition of yielding has been fulfilled. The curve 1 in Fig. 2 is the instantaneous straining diagram $\sigma = \varphi(p)$,

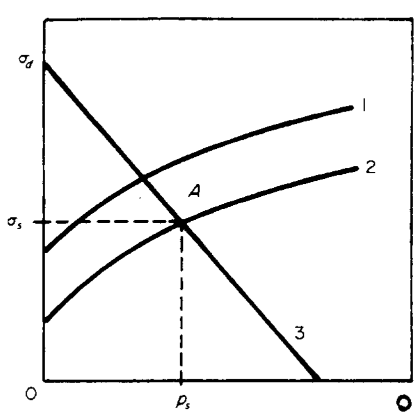

Fig. 2.

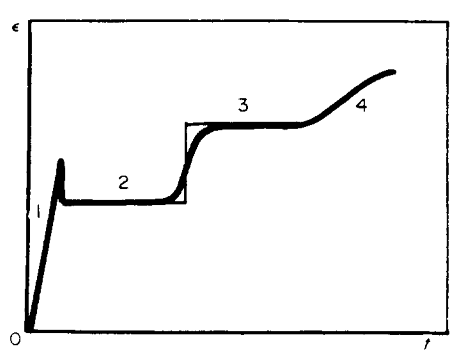

Fig. 3.

plotted in coordinates $\sigma \sim p$. Curve 2 is obtained by parallel shift of the curve I downwards by the value of $G_1(\sigma_d/VE)$. The equation of straight line 3 is $\sigma + \lambda Ep = \sigma_d$. The point of intersection A of curve 2 and straight line 3 determines the observed lower yield stress σ_s and respective instantaneous plastic strain. In this case too, it is necessary to solve the integral equation to construct the stress–strain curve. If $\dot{\sigma}$ = const, $\lambda = 0$, and eqn (6) remains valid. This analysis is true for element (for a specimen with a very short working part).

If a long specimen is stressed, the plastic strain front propagates from one end to the other. This process is thoroughly studied in works[8, 15]. The strain gauge fixed in some intermediate section indicates the following (Fig. 3): at first the specimen stretches uniformly remaining elastic, which is shown by the part I in the diagram. The maximum elastic strain at the end determines the peak stress, i.e. the upper yield stress σ_d. At $t = t_0$ the material yield delay capacity becomes exhausted, the stress drops down to value σ_s–the observed lower yield stress, and the strain remains elastic until the plastic front reaches the section (part 2). As soon as the plastic front reaches the gauge, the strain increases up to the value p and remains constant until the front reaches the other end of the specimen (part 3). After that the specimen continues deforming uniformly with strain hardening (part 4). The plastic strain front registered by the gauge turns out to be lengthened. This can be explained by the inclination of the plastic strain front, which, as indicated in [8] makes an angle approximating $\pi/4$ to the specimen axis, whereas the narrow gauge was fixed on the circumference.

The analysis of the process may be as follows. This is stress concentration (with coefficient k_1) at the specimen end near the grips (let it be section $x = 0$). Therefore the condition of delaying will be fulfilled primarily here. The stress drop at the end leads to the total stress drop down to value σ_s. The movement of plastic front begins from section $x = 0$ and is regulated by the condition of delaying. In some section x the plastic strain appears at the time when the condition of delaying is fulfilled in this section; that is to be written as follows

$$\int_0^{t_0} g(\sigma)\,\mathrm{d}t + \int_{t_0}^{t} g(\sigma_s)\,\mathrm{d}t + \int_t^{t+\Delta t} g(k_2\sigma_s)\,\mathrm{d}t = \tau_0$$

The last integral is determinated by concentration k_2 at the front of the moving plastic wave. Let us take the function of delaying as $g(\sigma) = (\sigma/\sigma_*)^n$ and assume $(\sigma_s/\sigma_*)^n = \tau_0/t_s$ where t_s is the delay time corresponding to constantly acting stress σ_s. Then, taking into account that $\Delta t = \Delta(\mathrm{d}t/\mathrm{d}x)$ (Δ is the section covered by the concentration) we obtain the following differential equation which describes the front movement

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k_2{}^n\Delta}{(1 - k_1{}^{-n})t_s + t_0 - t}\ .$$

Neglecting the value of $k_1{}^{-n}$ which is small compared with a unit and designating $\delta = k_2{}^n\Delta$, we obtain the following solution of this equation

$$x = -\delta \ln\left[1 - \frac{t - t_0}{t_s}\right]$$

which fulfils the condition $x = 0$ at $t = t_0$. If we take into account that $t_s \ll t_0$ and $t_s \ll t$, then

$$x \approx \delta(t - t_0)/t_s\ . \tag{9}$$

Expression (9) determines the constant velocity of the front movement; this conclusion is confirmed by experiments of work[8].

In addition to the designation $G_1(t)$ introduced earlier, let us assume

$$G(t) = \int_0^t K(t - s)\,\mathrm{d}s\,.$$

Then the constitutive equation in some section through which the plastic front has already passed will be written

$$\varphi(p) = \sigma_s[1 + G(t) - G(t_0)] + \sigma_d G_1(t_0)/t_0. \tag{10}$$

If the velocity of the machine grips movement is constant, then the loading rate in the elastic range is also constant and $\sigma = \eta t$, whereas $\sigma_d = \eta t_0$. Assuming in (10) $t = t_0$, we obtain

$$\varphi(p) = \sigma_s + \eta G_1(\sigma_d/\eta). \tag{11}$$

This expression connects the lower yield stress σ_s and length of plateau p. The second relation between them may be derived from the kinematic condition which connects grips velocity V with the front movement velocity

$$V = (xp)^{\cdot}. \tag{12}$$

To correlate (12) with expression (9) determining the front constant velocity, we should assume $\dot{p} = 0$. Then

$$V = \dot{x}p. \tag{13}$$

Comparing (9), (13) and taking into account definition of t_0, we obtain

$$\sigma_s = \sigma_*\left(\frac{V\tau_0}{\delta p}\right)^{1/n}. \tag{14}$$

Solving (11) and (14) together, we can find p and σ_s. The grafic solution diagram remains the same as in the analysis of the element, but the straight line is replaced by curve (14) of the hiperbolical type.

Having assumed $\dot{p} = 0$, we should believe that the time depending term in equation (10) is also negligible at $t > t_0$. This condition dictates selection of kernel type. For example, for Abel's kernel $K(t-s) = k(t-s)^{-\alpha}$ difference $G(t) - G(t_0)$ at any t may be made as small as one wants, provided that the selected value of α is sufficiently close to 1.

If we take a kernel as Abel's type and assume function $\varphi(p)$ is linear, $\varphi(p) = \varphi(0) + m^{-1}p$ we obtain from (11) the dependence between the upper and lower yield stresses

$$\varphi(0) + \frac{1}{m}p - \sigma_s = A/\sigma_d^{n(1-\alpha)-1}. \tag{15}$$

This dependence was verified by experiments of work[8] made on steel 50. The following parameters were calculated for this steel: $\alpha = 0.9$; $k = 0.02$; $\varphi(0) = 57.8$. The elastic strain rates measured on the specimen before appearance of the yielding in three groups of experiments were equal to 6×10^{-6}; 1.1×10^{-4} and 5×10^{-3} sek^{-1}. The slight scatter was observed in each group of tests corresponding to the given rate. The average value of η was equal to 21. The exponent at σ_d in (15) $n(1-\alpha) - 1 = 1.1$ differs but slightly from a unit. Figure 4 shows dependence of actually measured value $\sigma_p = \varphi(0) + m^{-1}p - \sigma_s$ on $1/\sigma_d$, the points are practically on the same straight line.

4. UNLOADING PROCESSES

All equations of state existing so far cannot satisfactory describe the process of unloading. According to the theory of Karman–Taylor–Rakhmatulin the unloading should be always elastic, whereas according to the theories of Malvern–Sokolovsky type that is viscous down to the static curve, neither of them is true. Therefore, usually independent equations are offered for description of the loading and unloading processes. The constitutive eqn (1) as distinct from the above-mentioned ones, makes it possible to describe the unloading[19] as well. If it begins at $t = t_*$, the value of $\dot{\sigma}$ becomes negative, the stress drops but the strain may continue growing for some time. Plastic strain cannot diminish, therefore the unloading yield stress should

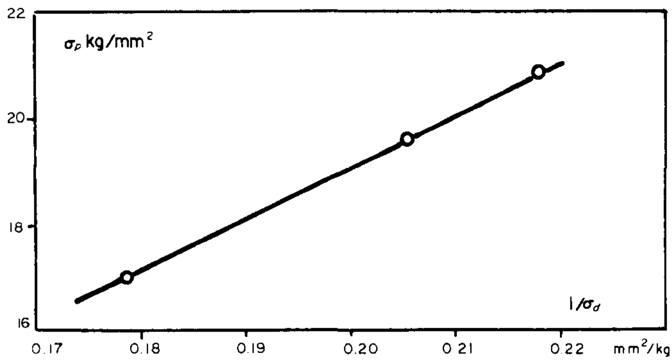

Fig. 4.

correspond to the extreme value of p, i.e. to condition $\dot{p} = 0$. Differentiating the equation (1) in time, we obtain

$$\varphi'(p)\dot{p} = \sigma + \frac{d}{dt}\int_0^t K(t-s)\sigma(s)\,ds = 0. \tag{16}$$

If the unloading low $\sigma(t)$ is known, from (16) we may determine both the instant of time t_{**} when the plastic strain reaches its maximum and the value of stress σ at this instant, i.e. the value of unloading yield stress. On reaching this stress further unloading is elastic.

If the process of loading was performed at some constant rate $\dot{\sigma}_l$ and unloading was almost at constant rate $\dot{\sigma}_u$, and if we select Abel's kernel as that the integral eqn (1), we obtain from (16)

$$\frac{\dot{\sigma}_l}{\dot{\sigma}_u}t^{1-\alpha} + \left(1 - \frac{\dot{\sigma}_l}{\dot{\sigma}_u}\right)(t - t_*)^{1-\alpha} + \frac{1-\alpha}{k} = 0. \tag{17}$$

Here, t_* is the instant the unloading beginning.

From (17) we can also determine the value of rate $\dot{\sigma}_u$ at which the unloading is elastic and viscous parts should not be observed. For this purpose we introduce $t = t_*$ in (17) and then

$$\dot{\sigma}_u = -\dot{\sigma}_l t_*^{1-\alpha}\frac{k}{1-\alpha}. \tag{18}$$

It is evident that this value depends both on the loading rate and on value t_*, i.e. on the value of the maximum stress (or strain) which was achieved by the moment of loading beginning. The analysis of the experimental data[19] indicated that, if the unloading rate is the same as loading, i.e. $\dot{\sigma}_u = -\dot{\sigma}_l$, or exceeds it, viscous effects are practically insignificant and cannot be discovered. However, if $\dot{\sigma}_u = -0.1\,\dot{\sigma}_l$, the viscous effects are essential. As an example, Fig. 5

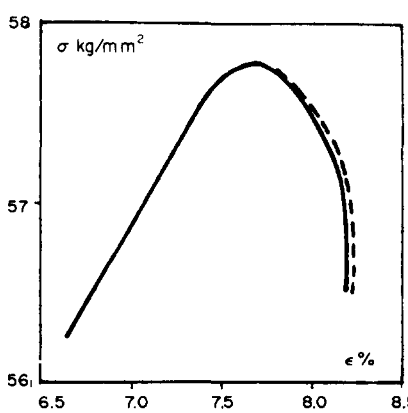

Fig. 5.

shows a part of the diagram $\sigma \sim \epsilon$ (continuous line) obtained at loading rate $\dot{\sigma}_l = -0.385\ \mathrm{kg/mm^2\,sek}$; the unloading rate was $\dot{\sigma}_u = -0.0385\ \mathrm{kg/mm^2 sek}$ [19]. The experiments were fulfilled on low-carbon steel, the kernel parameters were following: $\alpha = 0.93$, $\kappa = k/(1-2)(2-\alpha) = 1.0$. The time of unloading transition to the elastic state was $t_{**} = t_* + 27.5$ sek in accordance with the eqn (17). The broken curve in Fig. 5 is calculated in accordance with (1).

5. CONDITIONS OF PLASTIC STRAINING

It is known, that in ordinary hereditary-type viscoelasticity all stress-strain curves with various loading programs have the same initial moduli which are equal to instantaneous curve modulus. The presence of pure elastic state with various elastic histories leads to appearance of distinctions in values of initial plasticity moduli, thus, each stress–strain curve in plasticity range is in fact characterised by two parameters—by the yield stress σ_0 and by the initial plasticity modulus $\mathrm{d}\sigma/\mathrm{d}p\,|_{t=t_0}$.

The yield stress determination and the description of the transition from elastic to plastic state were given in Section 3. It is seen, that $\sigma_0 < \varphi(0)$, therefore the possibility of repeated elastic straining, i.e. the reversed transition from plastic to elastic deformation is reserved at some straining conditions. If at any time $t = t_*$ ($\sigma = \sigma_*$, $p = p_*$), the instantaneous additional loading is realised, the material behaviour can be described by new instantaneous curve, which corresponds to p_*. Let $\psi(p) = \varphi(p) - \varphi(0)$, then the equation of the new instantaneous curve will be

$$\varphi^*(p) = \varphi^*(p_*) + \psi(p)$$

where

$$\varphi^*(p_*) = \varphi(0) - \sigma_0 + \sigma_*.$$

It is seen from eqn (1), that the slope of the real stress–strain curve, i.e. $\mathrm{d}\sigma/\mathrm{d}p$ is always less than the slope of the instantaneous curve in the same point p:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}p} = \varphi'(p) - \frac{\mathrm{d}}{\mathrm{d}p} K^*\sigma$$

If we have the ordinary viscoelasticity without pure elastic deformation, then, as is mentioned before, $\varphi'(0) = \sigma'(0)$. In this case the condition $\mathrm{d}(K^*\sigma)/\mathrm{d}p \geq 0$ characterises the hardening process and is always fulfilled. In our present case when there is the pure elastic strain in the beginning, the initial plasticity moduli of the curves are not equal. Therefore it is natural to accept for the condition of plastic straining the following:

$$\frac{\mathrm{d}}{\mathrm{d}p} K^*\sigma \geq \frac{\mathrm{d}}{\mathrm{d}p} K^*\sigma \bigg|_{t=t_0}. \tag{19}$$

When the condition (19) is disturbed, the material will be deformed in accordance with the new instantaneous curve, which has been determined before. The appearance of additional elastic strain is now possible.

Performing differentiation in (19) and taking into account properties of the integral operator K^*, we obtain

$$\begin{aligned}\dot{\sigma} \leq \left\{\frac{\varphi'(p)}{\varphi'(0)}\left[\int_0^{t_0} K(t-s)\,\dot{\sigma}(s)\,\mathrm{d}s + \dot{\sigma}_0\right]\right. \\ \times\left.\left[\int_0^{t_0} K(t-s)\,\dot{\sigma}(s)\,\mathrm{d}s\right]^{-1} - 1\right\}\int_0^{t} K(t-s)\,\dot{\sigma}(s)\,\mathrm{d}s.\end{aligned} \tag{20}$$

Here, $\dot{\sigma}_0$ is the value of $\dot{\sigma}$ at the moment $t = t_0$. Condition (20) offers limitation by value $\dot{\sigma}$ with which the process may be conducted, remaining the plastic range.

The limitations by the value $\dot{\sigma}$ also makes it possible to explain the propagation of incremental waves with the elastic rate. Rates of additional loads, as a rule, are highly essential and exceed the permissible value of $\dot{\sigma}$.

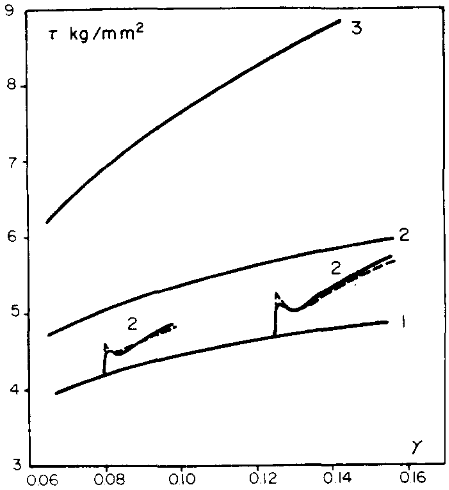

Fig. 6. (1) $\dot{\gamma} = 5\ 10^{-5}\ \text{sec}^{-1}$; (2) $\dot{\gamma} = 850\ \text{sec}^{-1}$; (3) $\varphi(\gamma)$—the instantaneous curve.

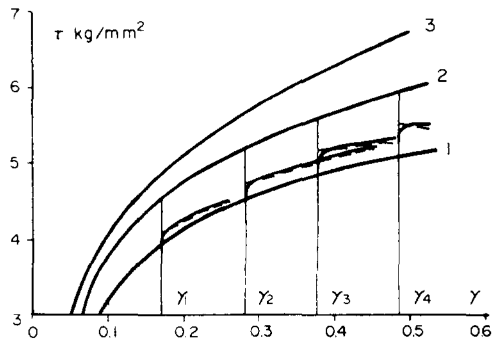

Fig. 7.

As an example we present the analysis of torsional experiments of Frantz and Duffy[21]. The velocity of the initial torsion was $\dot{\gamma}_1 = 5 \times 10^{-5}\ \text{sek}^{-1}$, the rate of increment $\dot{\gamma}_2 = 850\ \text{sek}^{-1}$. It turned out that a part of the incremental wave propagates at an elastic velocity; a new effect was also discovered—drop of stresses on reaching the yield stress during additional loading (see Fig. 6). If we take into account that additional loading is carried out at a constant rate and begins at some instant $t = t_*$ then for rate $\dot{\sigma}$ at the moment when the material changes to plastic state, i.e. at $t = t_{**}$, we obtain

$$\dot{\sigma}\left(1 + \frac{\varphi'(p_*)}{E}\right) = \dot{\varepsilon}\left[\varphi'(p_*) - E\frac{k}{1-\alpha}(t_{**} - t_*)^{1-\alpha}\right] - k\frac{\sigma(t_*)}{(t_{**} - t_*)^{\alpha}}. \tag{21}$$

It is seen from this expression that at certain material parameters and loading rates, $\dot{\sigma}$ may be a negative value. Figure 6 shows experiments of Frantz and Duffy and results of calculations (broken lines). Application of the above model with due regard for condition (19) to the description of repeated loading processes can be found in [20]. The subsequent analysis made it possible to explain results obtained by various authors which at first sight seem contradictory. (For example, work[14] proves that the history effect is not significant and that a new stress–strain diagram of repeated loading coincides with that which corresponds to the repeated loading rate. However, these data are at variance with, for example, results of [22, 23] which underline the substantial history effect.) It turns out that so different at first sight behaviour depends only on the material hereditary parameters and can be described within the bounds of one model.

Figure 7 shows experiments of Klepaczko[23] performed by torsion of aluminium bars at velocities $\dot{\gamma}_1 = 1.66 \times 10^{-5}\ \text{sek}^{-1}$ and $\dot{\gamma}_2 = 6.24 \times 10^{-1}\ \text{sek}^{-1}$. The initial loading took place at rate $\dot{\gamma}_2$ (curve 2) to value γ_1, γ_2, γ_3 and γ_4, then unloading and repeated loading at low rate $\dot{\gamma}_1$ (curve 1). Curve 3 in Fig. 7 is calculated instantaneous curve. The broken lines show calculated curves of the repeated loading.

At the moment when plastic strains appear, i.e. on the yield stress, rate $\dot{\sigma}$ on the right of it is determined by expression (21) from which it follows that $\dot{\sigma}$ for the same rate of repeated loading is the lesser the greater the value of strain at which it proceeds, since in this case φ' decreases, whereas the value $\sigma(t_*)$ increases.

6. WAVE PROCESSES

When investigated the wave processes in materials described by the offered model, it is necessary to take into account the existance of two constitutive equations: while the material is elastic, it is Hook low, when the material comes to the plastic stage, it is eqn (1). In the elastic range the solution is to be found by usual methods of the elasticity theory, in the plastic range it can be constructed by summing up solutions of relaxation and common visco-elasticity problems.

6.1 *The first problem*

The problem of relaxation is a solution of the following equation

$$\varphi(0) = \sigma_1(t) + \int_0^t K(t-s)\sigma_1(s)\,ds. \quad (22)$$

The left-hand part of this integral equation contains a constant value—the yield stress of the instantaneous straining curve. The eqn (22) may be rewritten as follows

$$\varphi(0) = \sigma_1(t) + \int_0^{t_0} K(t-s)\sigma_1(s)\,ds + \int_{t_0}^t K(t-s)\sigma_1(s)\,ds.$$

It is bound with the fact that at $t \leq t_0$ the material is elastic and variation law $\sigma_1(t)$ is known. Then the first integral may be calculated and for determination of $\sigma_1(t)$ we obtain the following integral equation

$$\sigma_1(t) + \int_0^{t_0} K(t-s)\sigma_1(s)\,ds = \varphi(0) - \int_0^{t_0} K(t-s)\sigma_1(s)\,ds.$$

It is indicated in [15] that if the elastic loading is carried out at a constant rate, then

$$\int_0^{t_0} K(t-s)\dot{\sigma}s\,ds = \frac{\sigma_s}{t_0}\{G_1(z+t_0) - G_1(z) - t_0 G_1'(z)\}$$

here

$$G_1(z) = \int_0^z K(z-\xi)\xi\,d\xi, \qquad z = t - t_0, \qquad \xi = s - t_0.$$

At great values of z function $G_1(z+t_0)$ may be expanded into series. Then, assuming $K^* = \kappa\mathscr{E}^*_{-\alpha}(-\beta)$, for determination of σ_1 we obtain the following integral equation

$$\sigma_1 + \kappa\mathscr{E}^*_{-\alpha}(-\beta)\sigma_1 = \varphi(0) - \frac{\sigma_s t_0}{2}\kappa\mathscr{E}_{-\alpha}(-\beta, z) + \ldots \quad (23)$$

Successive terms in expansion are not significant for the investigation of the assymptotic behaviour of the functions.

The eqn (23) may be solved by means of Laplace transformation

$$\sigma_1(t) \sim \frac{\varphi(0)}{1+\kappa/\beta}\left\{1 - \frac{1}{\beta\Gamma(\alpha)}\frac{1}{t^{1-\alpha}}\right\}.$$

It is clear that at great times the stress tends to some constant value which does not depend on the loading rate.

6.2 *The second problem*

The second problem is a solution of the problem for a medium with the following constitutive equation

$$\psi(p) = \sigma_2(t)_+ \int_0^t K(t-s)\sigma_2(s)\,ds \quad (24)$$

where $\psi(0) = 0$ and $\sigma_2(t) = 0$ at $t \leq t_0$.

As an example, we give a solution of the stress wave propagation problem in a semi-infinite bar[24]. The equation of motion with due regard for (24) may be written as follows

$$c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}-\int_0^t K(t-s)\frac{\partial^2 u}{\partial s^2}\,ds=0. \tag{25}$$

Here, u is displacement, $c^2=\varphi'(0)/\rho$.

Let us take the boundary and initial conditions as

$$u=\frac{\partial u}{\partial t}=0 \quad \text{at} \quad t=0$$
$$\frac{\partial u}{\partial t}=V_0 \quad \text{at} \quad x=0. \tag{26}$$

Then, assuming $K(t-s)=\mathscr{E}_{-\alpha}(-\beta, t-s)$ and applying Laplace transformation to (25) and (26), we obtain

$$c^2\frac{\partial^2\bar{u}}{\partial x^2}-p^2\bar{u}-\frac{\kappa}{p^{1-\alpha}+\beta}p^2\bar{u}=0. \tag{27}$$

Solution of the eqn (27)

$$\bar{u}=\frac{V_0}{p^2}\exp\left(-\frac{x}{c}\sqrt{p^2+\frac{p^2\kappa}{p^{1-\alpha}+\beta}}\right).$$

To find the inverse transformation, it is necessary to investigate behaviour of $\bar{u}$ in the area of singular points. Since the singular point determined by expression $1+\kappa/(p^{1-\alpha}+\beta)=0$, $(\beta>0$, $\kappa>0$, $0>\alpha>1)$ has argument $\varphi=\pi/(1-\alpha)$, it does not get in the plane with a slot along the negative real axis. Only one singular point $p=0$ remains to be investigated. In its area

$$\left(p^2+\frac{p^2\kappa}{p^{1-\alpha}+\beta}\right)^{1/2}=p+\sum_{m=0,\,K=1}^{m+K=n}\binom{1/2}{K}\kappa^K(-\beta)^m p^{1-(1-\alpha)(m+K)}+$$
$$+\sum_{m+K=n+1}^{\infty}\binom{1/2}{K}\kappa^K(-\beta)^m\frac{1}{p^{(1-\alpha)(m+K)-1}}, \tag{28}$$

Here, the number n is determined from condition

$$n(1-\alpha)-1<0, \quad (n+1)(1-\alpha)-1\geq 0.$$

Accordingly, we can present $\bar{u}=\bar{u}_1\cdot\bar{u}_2$, where

$$\bar{u}_1=\frac{V_0}{p}e^{S_1}, \qquad \bar{u}_2=\frac{1}{p}e^{S_2}$$
$$S_1=-\frac{x}{c}\left[p+\sum_{m+K=n+1}^{\infty}\binom{1/2}{K}\kappa^K(-\beta)^m\frac{1}{p^{(1-\alpha)(m+K)-1}}\right]$$
$$S_2=-\frac{x}{c}\sum_{m=0,\,K=1}^{m+K=n}\binom{1/2}{K}\kappa^K(-\beta)^m p^{1-(1-\alpha)(m+K)}$$

or

$$\bar{u}_1=\frac{V_0}{p}e^{-(x/c)p}\left\{1-\sum_{r=1}^{\infty}\sum_{m+K=n+1}^{\infty}\frac{(-1)^{r+1}}{r!}\left[\binom{1/2}{K}\kappa^K(-\beta)^m\times\frac{x}{c}\frac{1}{p^{(1-\alpha)(m+K)-1}}\right]^r\right\}$$
$$\bar{u}_2=\frac{1}{p}\left\{1-\sum_{r=1}^{\infty}\sum_{m=0,\,K=1}^{m+K=n}\frac{(-1)^{r+1}}{r!}\left[\binom{1/2}{K}\kappa^K(-\beta)^m\frac{x}{c}p^{1-(1-\alpha)(m+K)}\right]^r\right\}.$$

To find the inverse transformation $u(x, t)$ ○—● $\bar{u}(x, p)$, it is possible to find at first $u_1(x, t)$ ○—● $\bar{u}_1(x, p)$ and $u_2(x, t)$ ○—● $\bar{u}_2(x, p)$ and then to use convolution.

Methods of transformation of asymptotic expansion make it possible to obtain for great value of p and for small values of $(t - x/c)$:

$$u_1 = V_0\left\{1 - \sum_{r=1}^{\infty} \sum_{m+K=r+1}^{\infty} \frac{(-1)^{r+1}}{r!} \frac{1}{\Gamma[r(1-\alpha)(m+K)-(r-1)]} \times \left[\binom{1/2}{K}\kappa^K(-\beta)^m\left(\frac{x}{c}\right)\left(t-\frac{x}{c}\right)^{(m+K)(1-\alpha)-1}\right]^r\right\}.$$

Also for small p and great t

$$u_2 = 1 - \sum_{r=1}^{\infty} \sum_{m=0,K=1}^{m+K=n} \frac{(-1)^{r+1}}{r!} \frac{1}{\Gamma[r(1-\alpha)(m+K)-(r-1)]} \times \left[\binom{1/2}{K}\kappa^K(-\beta)^m\left(\frac{x}{c}\right)^{(m+K)(1-\alpha)}\right]^r.$$

Here, it is taken into account that since the solution is being looked for in the front area, $t \approx x/c$. Using convolution and differentiating the obtained expression in time for determination of $V = \partial u/\partial t$, we obtain

$$V = V_0 \sum_{m+K=n+1}^{\infty} \varphi\left[(1-\alpha)(m+K)-1, 1; Z\left(t-\frac{x}{c}\right)\right] \times \sum_{r=0}^{\infty} \sum_{m=0,K=1}^{m+K=n} \frac{1}{n!\Gamma[r((1-\alpha)(m+K)-1)+1]} Z^r\left(\frac{x}{c}\right) \tag{29}$$

where

$$Z(y) = -\binom{1/2}{K}\kappa^K(-\beta)^m \frac{x}{c}\, y^{(m+K)(1-\alpha)-1}$$

and φ is a function of Wright.

Specific cases of expression (29) at various values of α and β are given in [24].

The compatability conditions make it possible to easily calculate the values of stresses and strains at the front.

6.3 *Construction of the solution*

Now we shall prove that the sum of solutions obtained in 6.1 and 6.2 is the solution sought for. To do this, it is necessary to prove that $\sigma = \sigma_1 + \sigma_2$ satisfies the equation of motion, boundary and initial conditions, as well as the constitutive eqn (1). It is evident that the first requirement is satisfied automatically. As far as the boundary conditions are concerned, they may be selected for each of the problems as needed as, for example, it is done in [25]. It now remains only to show that σ satisfies the eqn (1). For this purpose we sum up (22) and (24). We obtain

$$\psi(p) + \varphi(0) = \sigma_1(t) + \sigma_2(t) + \int_0^t K(t-s)\sigma_1(s)\,ds + \int_0^t K(t-s)\sigma_2(s)\,ds$$

or

$$\varphi(p) = \sigma(t) + \int_0^t K(t-s)\sigma(s)\,ds.$$

Thus, solution of wave problems for media modelled by the eqn (1) actually reduces to solution of hereditary visco-elasticity problems.

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